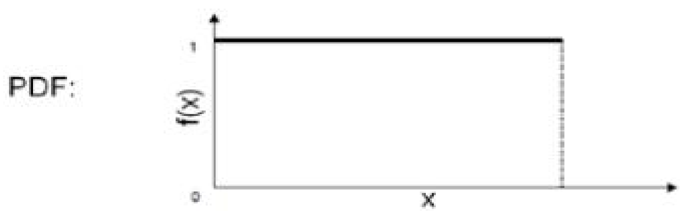
UNIT 5:Random number generation And Variation Generation

RANDOM-NUMBER GENERATION Random numbers are a necessary basic ingredient in the simulation of almost all discrete systems. Most computer languages have a subroutine, object, or function that will generate a random number. Similarly simulation languages generate random numbers that are used to generate event times and other random variables.

5.1 Properties of Random Numbers A sequence of random numbers, R1, R2... must have two important statistical properties, uniformity and independence. Each random number *Ri*, is an independent sample drawn from a continuous uniform distribution between zero and 1.
That is, the pdf is given by

$$\text{pdf:}\quad f(x) = \begin{cases} 1\,, & 0 \le x \le 1 \\ 0\,, & \text{otherwise} \end{cases}$$

The density function is shown below:



The expected value of Ri, is

$$E(R) = \int_0^1 x\,dx = [x^2/2]_0^1 = 1/2$$

The variance is given by 0

$$V(R) = \int_0^1 x^2 dx - [E(R)]^2$$

$$= [x^3/3]_0^1 - (1/2)^2 = 1/3 - 1/4$$

$$= 1/12$$

Some consequences of the uniformity and independence properties are the following:

1. If the interval (0, 1) is divided into n classes, or subintervals of equal length, the expected number of observations m each interval ii N/n where A' is the total number of observations.

2. The probability of observing a value in a particular interval is of the previous values drawn.

5.2 Generation of Pseudo-Random Numbers

Pseudo means false, so false random numbers are being generated. The goal of any generation scheme, is to produce a sequence of numbers between zero and 1 which simulates, or initiates, the ideal properties of uniform distribution and independence as closely as possible. When generating pseudo-random numbers, certain problems or errors can occur. These errors, or departures from ideal randomness, are all related to the properties stated previously. **Some examples include the following**

1) The generated numbers may not be uniformly distributed.

2) The generated numbers may be discrete -valued instead continuous valued

3) The mean of the generated numbers may be too high or too low.

4) The variance of the generated numbers may be too high or low

5) There may be dependence.

The following are examples:

a) Autocorrelation between numbers.

b) Numbers successively higher or lower than adjacent numbers.

c) Several numbers above the mean followed by several numbers below the mean.

Usually, random numbers are generated by a digital computer as part of the simulation. Numerous methods can be used to generate the values. In selecting among these methods, or routines, there are a number of important considerations.

1. The routine should be **fast.** The total cost can be managed by selecting a computationally efficient method of random-number generation.

2. The routine should be **portable** to different computers, and ideally to different programming languages .This is desirable so that the simulation program produces the same results wherever it is executed.

3. The routine should have a sufficiently **long cycle.** The cycle length, or period, represents the length of the random-number sequence before previous numbers begin to repeat themselves in an earlier order. Thus, if 10,000 events are to be generated, the period should be many times that long.

A special case cycling is degenerating. A routine degenerates when the same random numbers appear repeatedly. Such an occurrence is certainly unacceptable. This can happen rapidly with some methods.

4. The random numbers should be **replicable.** Given the starting point (or conditions), it should be possible to generate the same set of random numbers, completely independent of the system that is being simulated. This is helpful for debugging purpose and is a means of facilitating comparisons between systems.

5. Most important, and as indicated previously, the generated random numbers should closely approximate the ideal statistical properties of **uniformity and independences**

5.3 Techniques for Generating Random Numbers

5.3.1 The linear congruential method

It widely used technique, initially proposed by Lehmer [1951], produces a sequence of integers, $X1$, $X2$,... between zero and m — 1 according to the following recursive relationship:

$$X_{i+1} = (aX_i + c) \bmod m, \quad i = 0, 1, 2.... \quad (7.1)$$

The initial value **X0** is called the seed, **a** is called the constant multiplier, **c** is the increment, and **m** is the modulus.

If $c \neq 0$ in Equation (7.1), the form is called the **mixed congruential method.** When $c = 0$, the form is known as the **multiplicative congruential method**.

The selection of the values for a, c, m and X0 drastically affects the statistical properties and the cycle length. An example will illustrate how this technique operates.

EXAMPLE 1 Use the linear congruential method to generate a sequence of random numbers with X_0 = 27, a= 17, c = 43, and m = 100.

Here, the integer values generated will all be between zero and 99 because of the value of the modulus. These random integers should appear to be uniformly distributed the integers zero to 99.

Random numbers between zero and 1 can be generated by

Ri =Xi/m, i= 1,2,…… (7.2)

The sequence of Xi and subsequent Ri values is computed as follows:

X0 = 27

X1 = (17*27 + 43) mod 100 = 502 mod 100 = 2 R1=2/100=0. 02

X2 = (17*2 + 43) mod 100 = 77 mod 100 = 77 R2=77 /100=0. 77

X3 = (17*77+ 43) mod 100 = 1352 mod 100 = 52 R3=52 /100=0. 52

Second, to help achieve maximum density, and to avoid cycling (i.e., recurrence of the same sequence of generated numbers) in practical applications, the generator should have the largest possible period. Maximal period can be achieved by the proper choice of a, c, m, and X0.

The max period (P) is:

- **For m a power of 2, say m = 2b, and c ¹≠0, the longest possible period is P = m = 2b,** which is achieved provided that c is relatively prime to m (that is, the greatest common factor of c and m is 1), and a = 1 + 4k, where k is an integer.
- **For m a power of 2, say m = 2b, and c = 0, the longest possible period is P = m / 4 = 2b-2,** which is achieved provided that the seed X0 is odd and the multiplier, a, is given by a = 3 + 8k or a = 5 + 8k, for some k = 0, 1,...
- **For m a prime number and c = 0, the longest possible period is P = m - 1**, which is achieved provided that the multiplier, a, has the property that the smallest integer k such that ak - 1 is divisible by m is
  k = m – 1.

Multiplicative Congruential Method:

Basic Relationship:

Xi+1 = a Xi (mod m), where a ≥ 0 and m ≥ 0 … (7.3)

Most natural choice for **m** is one that equals to the capacity of a computer word. m = 2b (binary machine), where b is the number of bits in the computer word.

m = 10d (decimal machine), where d is the number of digits in the computer word.

EXAMPLE 1: Let m = 102 = 100, a = 19, c = 0, and X0 = 63, and generate a sequence c random integers using Equation

Xi+1 = (aXi + c) mod m, i = 0, 1, 2....

X0 = 63 X1 = (19)(63) mod 100 = 1197 mod 100 = 97

X2 = (19) (97) mod 100 = 1843 mod 100 = 43

X3 = (19) (43) mod 100 = 817 mod 100 = 17 . . . .

When m is a power of 10, say m = 10b, the modulo operation is accomplished by saving the b rightmost (decimal) digits.

5.3.2 Combined Linear Congruential Generators

As computing power has increased, the complexity of the systems that we are able to simulate has also increased. One fruitful approach is to combine two or more multiplicative congruential generators in such a way that the combined generator has good statistical properties and a longer period. The following result from L'Ecuyer [1988] suggests how this can be done: If $W_{i,1}$, $W_{i,2}$,... , $W_{i,k}$ are any independent, discrete-valued random variables (not necessarily identically distributed), but one of them, say $W_{i,1}$, is uniformly distributed on the integers 0 to m_i — 2, then

$$W_i = \left(\sum_{j=1}^{k} (-1)^{j-1} W_{i,j} \right) \bmod m_1 - 1$$

is uniformly distributed on the integers 0 to m_i — 2. To see how this result can be used to form combined generators, let $X_{i,1}$, $X_{i,2}$,..., $X_{i,k}$ be the i *th output from k different multiplicative congruential generators, where the* j *th generator has prime modulus m_j, and the multiplier a_j is chosen so that the period is m_j — 1. Then the j'th generator is producing integers $X_{i,j}$ that are approximately uniformly distributed on 1 to m_j - 1, and $W_{i,j} = X_{i,j} — 1$ is approximately uniformly distributed on 0 to m_j - 2. L'Ecuyer [1988] therefore suggests combined generators of the form*

$$Xi = \left(\sum_{j=1}^{k} (-1)^{j-1} X_{i,j} \right) \bmod m_1 - 1$$

$$Ri = \begin{cases} \dfrac{X_i}{m_1}, X_i > 0 \\ \dfrac{m_1 - 1}{m_1}, X_i = 0 \end{cases}$$

Notice that the "$(-1)^{j-1}$" coefficient implicitly performs the subtraction $X_{i,1}-1$; for example, if k = 2, then

$$(-1)^0(X_{i,1}-1)-(-1)^1(X_{i,2}-1)=\sum_{j=1}^{2}(-1)^{j-1}X_{i,j}$$

The maximum possible period for such a generator is

$$p = \frac{(m_1-1)(m_2-1)\dots(m_k-1)}{2^{k-1}}$$

5.4 Tests for Random Numbers

1. ***Frequency test***. Uses the Kolmogorov-Smirnov or the chi-square test to compare the distribution of the set of numbers generated to a uniform distribution.
2. ***Autocorrelation test***. Tests the correlation between numbers and compares the sample correlation to the expected correlation of zero.

5.4.1 Frequency Tests

A basic test that should always be performed to validate a new generator is the test of uniformity. Two different methods of testing are available.

1. Kolmogorov-Smirnov(KS test) and

2. Chi-square test.

• Both of these tests measure the degree of agreement between the distribution of a sample of generated random numbers and the theoretical uniform distribution.

• Both tests are on the null hypothesis of no significant difference between the sample distribution and the theoretical distribution.

1. The Kolmogorov-Smirnov test. This test compares the continuous cdf, F(X), of the uniform distribution to the empirical cdf, SN(x), of the sample of N observations. By definition,

F(x) = x, 0 ≤ x ≤ 1

If the sample from the random-number generator is R1 R2, ,..., RN, then the empirical cdf, SN(x), is defined by

$$S_n(x)\frac{\text{number of R1, R2, \ldots, Rn which are} \leq \text{x}}{N}$$

*The Kolmogorov-Smirnov test is based on the largest absolute deviation between F(x) and SN(X) over the range of the random variable. That is. it is based on the statistic **D = max |F(x) -SN(x)|** For testing against a uniform cdf, the test procedure follows these steps:*

***Step 1:** Rank the data from smallest to largest. Let R (i) denote the i th smallest observation, so that*

$$R(1) \leq R(2) \leq \ldots \leq R(N)$$

***Step 2:** Compute*

$$D^+ = \max_{1\le i\le n}\left\{\frac{i}{N} - R_{(i)}\right\}$$

$$D^- = \max_{1\le i\le n}\left\{R_{(i)} - \frac{i-1}{N}\right\}$$

Step 3: *Compute D = max (D+, D-).*

Step 4: *Determine the critical value,* ***Dα,*** *from* ***Table A.8*** *for the specified significance level α and the given sample size N.*

Step 5:

$D \le D_\alpha$ Accept: No Difference between **S_N(x) and F(x)**

$D > D_\alpha$ Reject: No Difference between **S_N(x) and F(x)**

We conclude that no difference has been detected between the true distribution of {R1, R2,... RN} and the uniform distribution.

EXAMPLE 6: *Suppose that the five numbers* ***0.44, 0.81, 0.14, 0.05, 0.93*** *were generated, and it is desired to perform a test for uniformity using the Kolmogorov-Smirnov test with a level of significance* ***α of 0.05.***

Step 1: *Rank the data from smallest to largest. 0.05, 0.14, 0.44, 0.81, 0.93*

Step 2: *Compute* ***D+ and D-***

	R_i	$\frac{i}{N}$	$D^+ = \max_{1\le i\le n}\left\{\frac{i}{N} - R_{(i)}\right\}$	$D^- = \max_{1\le i\le n}\left\{R_{(i)} - \frac{i-1}{N}\right\}$
1	0.05	0.20	0.15	0.05
2	0.14	0.40	0.26	~
3	0.44	0.60	0.16	0.04
4	0.81	0.80	~	0.21
5	0.93	1.00	0.07	0.13

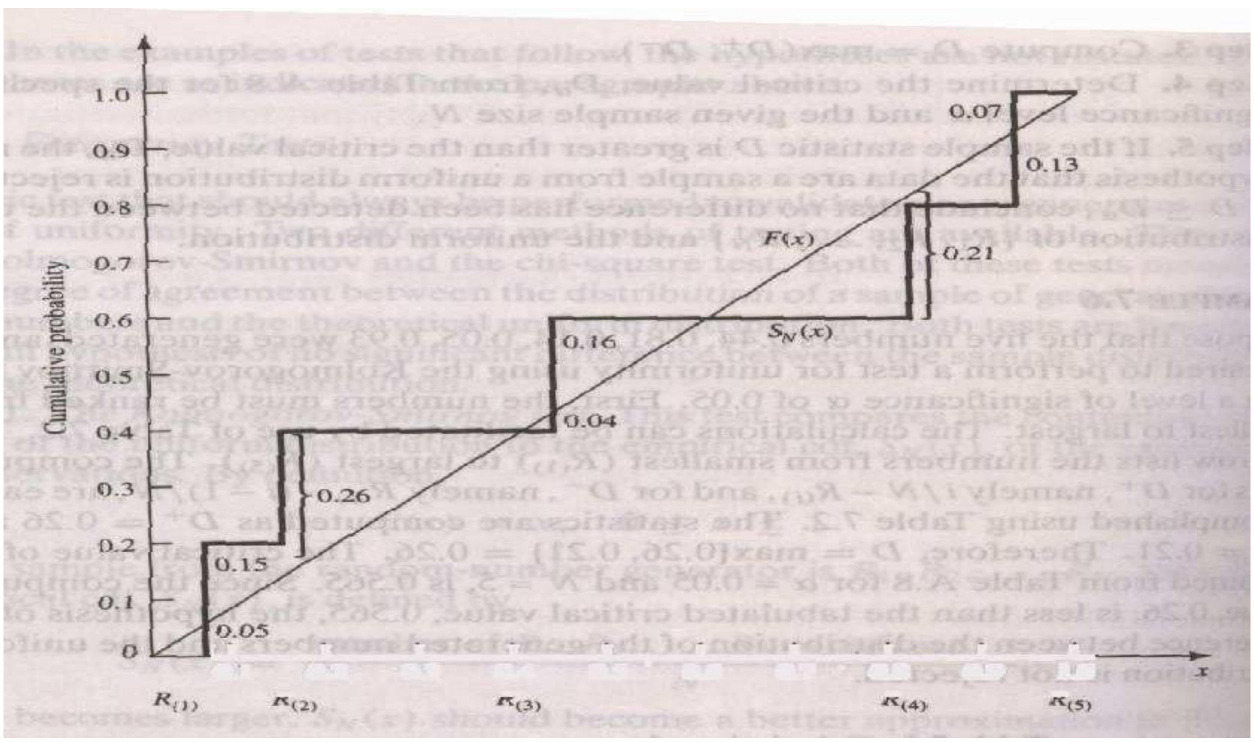
Step3: Compute D = max (D+, D-)

.   D=max (0.26, 0.21) = 0.26

Step 4: Determine the critical value, Dα, from Table A.8 for the specified significance level α and the given sample size N. **Here α=0.05, N=5 then value of Dα = 0.565**

Step 5: Since the computed value, 0.26 is less than the tabulated critical value, 0.565,

the hypothesis of no difference between the distribution of the generated numbers and the uniform distribution is not rejected.



compare F(x) with Sn(X)

2. The chi-square test.

The chi-square test uses the sample statistic

$$\chi_0^2 = \sum_{i=0}^{n} \frac{(O_i - E_i)^2}{E_i}$$

Where, Oi is observed number in the i th class

Ei is expected number in the i th class,

$$E_i = \frac{N}{n}$$

N – No. of observation

n – No. of classes

Note: sampling distribution of χ_0^2 approximately the chi square has n-1 degrees of freedom

Example 7: *Use the chi-square test with α = 0.05 to test whether the data shown below are uniformly distributed. The test uses n = 10 intervals of equal length, namely [0, 0.1), [0.1, 0.2)... [0.9, 1.0).* ***(REFER TABLE A.6)***

0.34	0.90	0.25	0.89	0.87	0.44	0.12	0.21	0.46	0.67
0.83	0.76	0.79	0.64	0.70	0.81	0.94	0.74	0.22	0.74
0.96	0.99	0.77	0.67	0.56	0.41	0.52	0.73	0.99	0.02
0.47	0.30	0.17	0.82	0.56	0.05	0.45	0.31	0.78	0.05
0.79	0.71	0.23	0.19	0.82	0.93	0.65	0.37	0.39	0.42
0.99	0.17	0.99	0.46	0.05	0.66	0.10	0.42	0.18	0.49
0.37	0.51	0.54	0.01	0.81	0.28	0.69	0.34	0.75	0.49
0.72	0.43	0.56	0.97	0.30	0.94	0.96	0.58	0.73	0.05
0.06	0.39	0.84	0.24	0.40	0.64	0.40	0.19	0.79	0.62
0.18	0.26	0.97	0.88	0.64	0.47	0.60	0.11	0.29	0.78

Interval	Range	O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
1	0.0-0.1	8	10	-2	4	0.4
2	0.1-0.2	8	10	-2	4	0.4
3	0.2-0.3	10	10	0	0	0.0
4	0.3-0.4	9	10	-1	1	0.1
5	0.4-0.5	12	10	2	4	0.4
6	0.5-0.6	8	10	-2	4	0.4
7	0.6-0.7	10	10	0	0	0.0
8	0.7-0.8	14	10	4	16	1.6
9	0.8-0.9	10	10	0	0	0.0
10	0.9-1.0	11	10	1	1	0.1
		100	100	0		3.4

The value of χ_0^2 is 3.4. This is compared with the critical value $\chi_{0.05,9}^2$ = 16.9.Since χ_0^2 is much smaller than the tabulated value of $\chi_{0.05,9}^2$, the null hypothesis of a uniform distribution is not rejected.

5.4.2 Tests for Auto-correlation

The tests for auto-correlation are concerned with the dependence between numbers in a sequence. The list of the 30 numbers appears to have the effect that every 5th number has a very large value. If this is a regular pattern, we can't really say the sequence is random.

0.12	0.01	0.23	0.28	0.89	0.31	0.64	0.28	0.83	0.93
0.99	0.15	0.33	0.35	0.91	0.41	0.60	0.27	0.75	0.88
0.68	0.49	0.05	0.43	0.95	0.58	0.19	0.36	0.69	0.87

The test computes the auto-correlation between every m numbers (m is also known as the lag) starting with the ith number. Thus the autocorrelation ρ_{im} between the following numbers would be of interest.

$$R_i, R_{i+m}, R_{i+2m}, \ldots, R_{i+(M+1)m}$$

Form the test statistic $Z_0 = \frac{\hat{\rho}_{im}}{\sigma_{\hat{\rho}_{im}}}$ which is distributed normally with a mean of zero and a variance of one.

The actual formula for $\hat{\rho}_{im}$ and the standard deviation is $\hat{\rho}_{im} = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+km} R_{(k+1)m}\right] - 0.25$ and

$$\sigma_{\hat{\rho}_{im}} = \frac{\sqrt{13M+7}}{12(M+1)}$$

After computing Z_0, do not reject the null hypothesis of independence if

$$-z_{\alpha/2} \le Z_0 \le z_{\alpha/2}$$

where α is the level of significance.

EXAMPLE : Test whether the 3rd, 8th, 13th, and so on, numbers in the sequence at the beginning of this section are auto correlated. (Use a = 0.05.) Here, i = 3 (beginning with the third number), m = 5 (every five numbers), N = 30 (30 numbers in the sequence), and M = 4 (largest integer such that 3 + (M +1)5 < 30).

0.12	0.01	0.23	0.28	0.89	0.31	0.64	0.28	0.83	0.93
0.99	0.15	0.33	0.35	0.91	0.41	0.60	0.27	0.75	0.88
0.68	0.49	0.05	0.43	0.95	0.58	0.19	0.36	0.69	0.87

Solution:

$$\rho_{i_m} = \frac{1}{4+1}\left[(0.23)(0.28)+(0.28)(0.33)+(0.33)(0.27)+(0.27)(0.05)+(0.05)(0.36)\right]-0.25$$
$$= -0.1945$$

And

$$\sigma_{\rho_{im}} = \frac{\sqrt{13(4)+7}}{12(4+1)} = 0.1280$$

Then, test for statistic assumes the value

$$Z_0 = -\frac{0.1945}{0.1280} = -1.516$$

Now the critical value from Table A.3 is $Z_{0.025}$=1.96

Therefore, the hypothesis of independence can't be rejected on the basis of this test.

2.Random Variate Generation TECHNIQUES:

- INVERSE TRANSFORMATION TECHNIQUE
- ACCEPTANCE-REJECTION TECHNIQUE

All these techniques assume that a source of uniform (0,1) random numbers is available R1,R2….. where each R1 has probability density function and cumulative distribution function.

Note: The random variable may be either discrete or continuous.

2.1 Inverse Transform Technique

The inverse transform technique can be used to sample from exponential, the uniform, the Weibull and the triangle distributions.

2.1.1 Exponential Distribution

The exponential distribution, has probability density function (pdf) given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & 0 \le x \\ 0, & x < 0 \end{cases}$$

and cumulative distribution function (cdf) given by

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \begin{cases} 1 - e^{-\lambda x}, & 0 \le x \\ 0, & x < 0 \end{cases}$$

The parameter λ can be interpreted as the mean number of occurrences per time unit. For example, if interarrival times X_1, X_2, X_3 . . . had an exponential distribution with rate, and then could be interpreted as the mean number of arrivals per time unit, or the arrival rate. For any i,

$$E(X_i) = 1/\lambda$$

And so 1/λ is mean inter arrival time. The goal here is to develop a procedure for generating values X1, X2, X3 . . . which have an exponential distribution.

The inverse transform technique can be utilized, at least in principle, for any distribution. But it is most useful when the cdf. F(x), is of such simple form that its inverse, F^{-1}, can be easily computed.

A step-by-step procedure for the inverse transform technique illustrated by me exponential distribution, is as follows:

Step 1: Compute the cdf of the desired random variable X. For the exponential distribution, the cdf is

$$F(x) = 1-e^{-\lambda x}, \; x \ge 0.$$

Step 2: Set F(X) = R on the range of X. For the exponential distribution, it becomes

$$1 - e^{-\lambda X} = R \text{ on the range } x \ge 0.$$

Since X is a random variable (with the exponential distribution in this case), so $1-e^{-\lambda x}$ is also a random variable, here called R. As will be shown later, R has a uniform distribution over the interval (0,1).,

Step 3: Solve the equation F(X) = R for X in terms of R. For the exponential distribution, the solution proceeds as follows:

$$1 - e^{-\lambda x} = R$$

$$e^{-\lambda x} = 1 - R$$

$$-\lambda X = \ln(1 - R)$$

$$x = -1/\lambda \ln(1 - R) \qquad \ldots(5.1)$$

Equation (5.1) is called a random-variate generator for the exponential distribution. In general, Equation (5.1) is written as $X=F^{-1}(R)$. Generating a sequence of values is accomplished through steps 4.

Step 4: Generate (as needed) uniform random numbers R1, R2, R3,... and compute the desired random variates by

$$X_i = F^{-1}(R_i)$$

For the exponential case, $\mathbf{F^{-1}(R) = (-1/\lambda)\ln(1- R)}$ by Equation (5.1),

so that $\mathbf{X_i = -1/\lambda \ln(1 - R_i) \ldots(5.2)}$ for i = 1,2,3,.... One simplification that is usually employed in Equation (5.2) is to replace 1 – Ri by Ri to yield $\mathbf{X_i = -1/\lambda \ln R_i \ldots(5.3)}$ which is justified since both Ri and 1- Ri are uniformly distributed on (0,1).

Example: consider the random number As fellows, where λ=1

R_i	0.1306	0.0422	0.6597	0.7965	0.7696

Solution:

Using equation compute Xi

$$x = -1/\lambda \ln(1 - R)$$

I	1	2	3	4	5
R_i	0.1306	0.0422	0.6597	0.7965	0.7696
X_i	0.1400	0.0431	1.078	1.592	1.468

Uniform Distribution :

Consider a random variable X that is uniformly distributed on the interval [a, b]. A reasonable guess for generating X is given by

$$X = a + (b - a)R \quad \text{……….5.5}$$

[Recall that R is always a random number on (0,1).

The pdf of X is given by

$$f(x) = \begin{cases} 1/(b-a), & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

The derivation of Equation (5..5) follows steps 1 through 3 of Section 5.1.1:

Step 1. The cdf is given by

$$F(x) = \begin{cases} 0, & x < a \\ (x - a)/(b - a), & a \le x \le b \\ 1, & x > b \end{cases}$$

Step 2. Set F(X) = (X - a)/(b -a) = R

Step 3. Solving for X in terms of R yields

$$X = a + (b - a)R,$$

which agrees with Equation (5.5).

Weibull Distribution:

The weibull distribution was introduce for test the time to failure of the machine or electronic components. The location of the parameters V is set to 0.

$$f(x) = \begin{cases} \frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-(x/\alpha)^\beta}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha>0$ and $\beta>0$ are the scale and shape of parameters.

Steps for Weibull distribution are as follows:

step 1: The cdf is given by

$$F(X) = 1 - e^{-(x/\alpha)^\beta}, x \geq 0.$$

step2 :set f(x)=R

$$1 - e^{-(X/\alpha)^\beta} = R.$$

step 3:Solving for X in terms of R yields.

$$X = \alpha[-\ln(1-R)]^{1/\beta}$$

Empirical continuous distribution:

Respampling of data from the sample data in systamtic manner is called empirical continuos distribution.

Step1:Arrange data for smallest to largest order of interval

$x(i-1)<x<X(i)$ i=0,1,2,3….n

Step2:Compute probability 1/n

Step3:Compute cumulative probability i.e i/n   where n is interval

step4:calculate a slope i.e

without frequency   ai=x(i)-x(i-1)/(1/n)

with frequency ai= x(i)-x(i-1)/(c(i)-c(i-1)   where c(i) is cumulative probability

2.1 Acceptance-Rejection technique

- Useful particularly when inverse cdf does not exist in closed form
- Illustration: To generate random variants, $X \sim U(1/4, 1)$
- Procedures:

Step 1: Generate a random number R ~ U [0, 1]

Step 2a: If R ≥ ¼, accept X=R.

Step 2b: If R < ¼, reject R, return to Step 1

- R does not have the desired distribution, but R conditioned (R') on the event {R ³ ¼} does.
- Efficiency: Depends heavily on the ability to minimize the number of rejections.

2.1.1 Poisson Distribution *A Poisson random variable, N, with mean a > 0 has pmf*

$$p(n) = P(N = n) = \frac{e^{-\alpha}\alpha^n}{n!}, \; n = 0, 1, 2, \dots$$

- N can be interpreted as number of arrivals from a Poisson arrival process during one unit of time
- Then time between the arrivals in the process are exponentially distributed with rate α.
- Thus there is a relationship between the (discrete) Poisson distribution and the (continuous) exponential distribution, namely

$$N = n \Leftrightarrow \sum_{i=1}^{n} A_i \le 1 < \sum_{i=1}^{n+1} A_i$$

$$\sum_{i=1}^{n} A_i \le 1 < \sum_{i=1}^{n+1} A_i \Leftrightarrow \sum_{i=1}^{n} -\frac{1}{\alpha}\ln R_i \le 1 < \sum_{i=1}^{n+1} -\frac{1}{\alpha}\ln R_i$$

$$\Leftrightarrow \prod_{i=1}^{n} R_i \ge e^{-\alpha} > \prod_{i=1}^{n+1} R_i$$

The procedure for generating a Poisson random variate, N, is given by the following steps:

Step 1: *Set n = 0, and P = 1*

Step 2: *Generate a random number Rn+1 and let P = P. Rn+1*

Step 3: *If P < $e^{-\alpha}$, then accept N = n. Otherwise, reject current n, increase n by one, and return to step 2*

Example: *Generate three Poisson variants with mean α =0.2 for the given Random number*

0.4357,0.4146,0.8353,0.9952,0.8004

Solution:

Step 1.Set n = 0, P = 1.

tep 2.R1 = 0.4357, P = 1 • R1 = 0.4357.

Step 3. Since P = 0.4357 < e^{-b} = 0.8187, accept N = 0. ***Repeat Above procedure***

n	R_{n+1}	p	accept/reject	Result
0	0.4357	0.4357	$P<e^{-\alpha}$ (accept)	N=0
0	0.4146	0.4146	$P<e^{-\alpha}$ (accept)	N=0
0	0.8353	0.8353	$P\geq e^{-\alpha}$ (reject)	
1	0.9952	0.8313	$P\geq e^{-\alpha}$ (reject)	
2	0.8004	0.6654	$p < e^{-\alpha}$ (accept)	N=2

Gamma distribution:

Is to check the random variants are accepted or rejected based on dependent sample data.

Steps 1: Refer the steps which given in problems.

Unit-5

Random Number Generation & Random Variate Generation

Problems:

(1) Generate a sequence of 5 integer Random number with $a = 19$, $m = 100$, $x_0 = 63$, $c = 1$ (mixed Linear Congruential Method)

Solution: $X_{i+1} = (a x_i + c) \bmod m$, $i = 0 \ldots m-1$

→ $i = 0$ $X_1 = (19 \times 63 + 1) \bmod 100$

$= 1198 \bmod 100 = 98$

$i = 1$ $R_i = \frac{x_i}{m}$, $i = 1 \ldots m$

$R_1 = \frac{98}{100} = 0.98$

→ $i = 1$ $X_2 = (19 \times 98 + 1) \bmod 100 = 63$

$i = 2$ $R_2 = \frac{63}{100} = 0.63$

→ $i = 2$ $X_3 = (19 \times 63 + 1) \bmod 100 = 98$

$i = 3$ $R_3 = 98/100 = 0.98$

→ $i = 3$ $X_4 = (19 \times 98 + 1) \bmod 100 = 63$

$i = 4$ $R_4 = 63/100 = 0.63$

→ $i = 4$ $X_5 = (19 \times 63 + 1) \bmod 100 = 98$

$i = 5$ $R_5 = 98/100 = 0.98$

∴ Random numbers are 0.98, 0.63, 0.98, 0.63, 0.98

(2) Using <u>Multiplicative Congruential Method</u>, Generate sequence of 5 integer number where $a = 24$, $m = 100$, seed value $= 64$.

<u>Solution:</u>

<u>Given</u>

$a = 24$

$m = 100$

$x_0 = 64$

$c = 0$

$$X_{i+1} = (aX_i + c) \bmod m, \quad i = 0 \ldots m-1$$

$$R_i = \frac{X_i}{m}, \quad i = 1 \ldots m$$

$X_1 = (24 \times 64 + 0) \bmod 100 = 36$

$R_1 = 36/100 = 0.36$

$X_2 = (24 \times 36 + 0) \bmod 100 = 64$

$R_2 = 64/100 = 0.64$

$X_3 = (24 \times 64 + 0) \bmod 100 = 36$

$R_3 = 36/100 = 0.36$

$X_4 = (24 \times 36 + 0) \bmod 100 = 64$

$R_4 = 64/100 = 0.64$

$X_5 = (24 \times 64 + 0) \bmod 100 = 36$

$R_5 = 36/100 = 0.36$

∴ Random numbers are 0.36, 0.64, 0.36, 0.64, 0.36

(3) Using **K.S Test** with $\alpha = 0.05$, to Test wether the data shown below are uniformly distributed 0.44, 0.81, 0.14, 0.05, 0.93 & also plot graph for $f(x)$.

Solution :

given, $\alpha = 0.05$
$N = 5$ (No. of random number given)

i	$R_{(i)}$	i/N	$\frac{i-1}{N}$	$D^+ = \max_{1 \le i \le N} \{ \frac{i}{N} - R_i \}$	$D^- = \max_{1 \le i \le N} \{ R_i - \frac{i-1}{N} \}$
1	0.05	0.20	0	0.15	0.05
2	0.14	0.40	0.20	**0.26** max.	~
3	0.44	0.60	0.40	0.16	0.04
4	0.81	0.80	0.60	~	**0.21** max.
5	0.93	1.00	0.80	0.07	0.13

$$D = \max(D^+, D^-)$$
$$= \max(0.26, 0.21) = \underline{0.26}$$

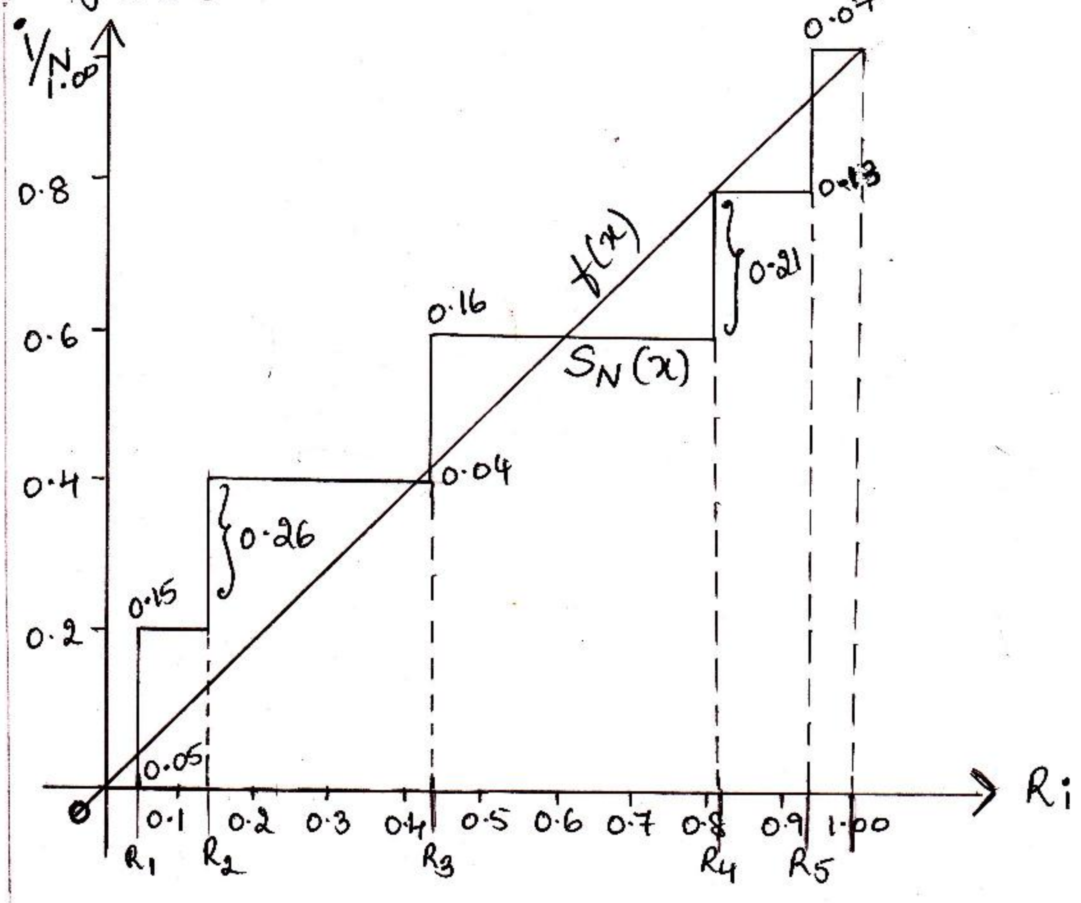
$D_{\alpha}, N \Rightarrow D_{0.05}, 5 = 0.565$, from table A8

$\therefore D \le D_\alpha \Rightarrow 0.26 \le 0.565$

$\therefore$ Accepted Null hypothesis

hence Random Numbers are uniformly distributed.

graph :



(4) Using K.S Test with $\alpha = 0.05$, & sample data are given below also plot graph for $f(x)$.

0.46, 0.18, 0.23, 0.64, 0.36, 0.44

Solution: given $\alpha = 0.05$, $N = 6$.

i	R_i	i/N	$\frac{i-1}{N}$	$D^+ = \max_{1\le i\le N}\{i/N - R_i\}$	$D^- = \max_{1\le i\le N}\{R_i - \frac{i-1}{N}\}$
1	0.18	0.16	0	~	0.18 max
2	0.23	0.33	0.16	0.1	0.07
3	0.36	0.5	0.33	0.14	0.03
4	0.44	0.66	0.5	0.22	~
5	0.46	0.83	0.66	0.37 max	~
6	0.64	1.00	0.83	0.36	~

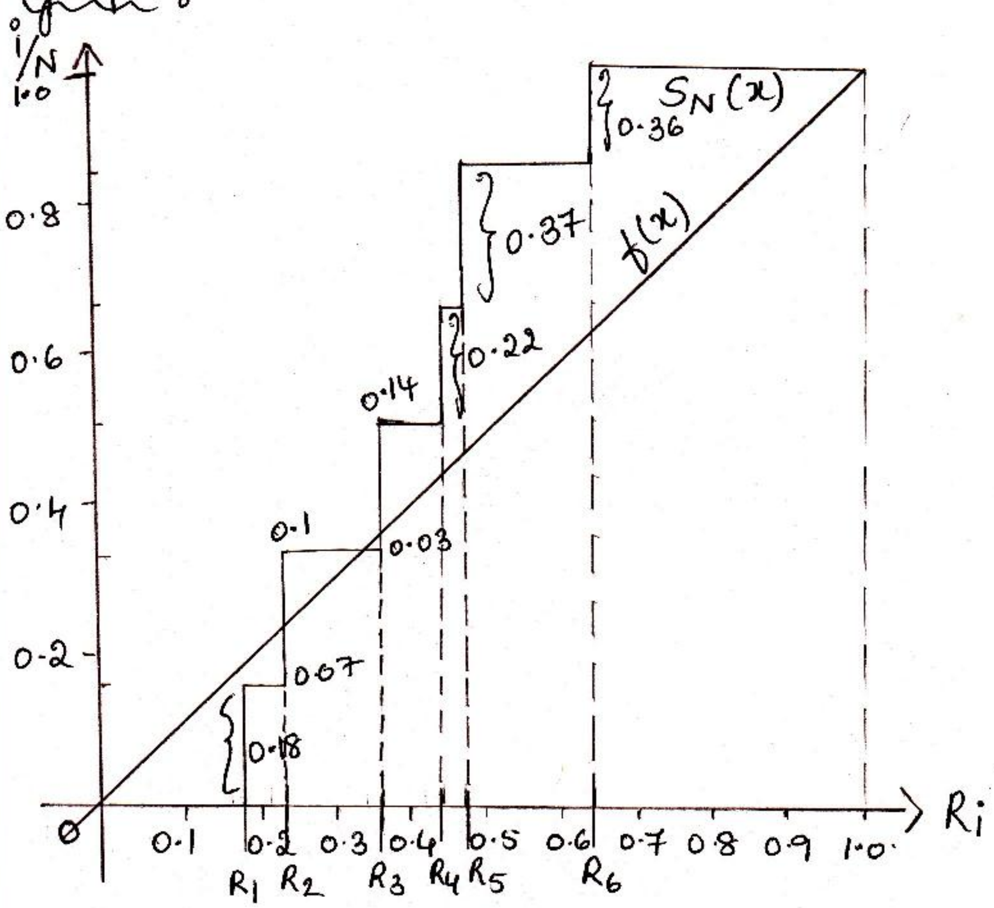
∴ $D = max(D^+, D^-) \Rightarrow max(0.37, 0.18) = 0.37$

from table A8, $D_{\alpha, N} = D_{0.05, 6} = 0.521$

∴ $D \leq D_\alpha = 0.37 \leq 0.521$

∴ Accepted Null hypothesis
hence random numbers are uniformly distributed.

Graph:



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(5) Using chi-square Test with level of significance $\alpha = 0.05$, Test wether given data are uniformly distributed. with test uses 10 interval of equal length. total number of sample data are 100.

*/. */. VVI

Interval	1	2	3	4	5	6	7	8	9	10
observation	8	8	10	9	12	8	10	14	10	11

Solution :

given, $\alpha = 0.05$, $n = 10$, $N = 100$

$$E_i = \frac{N}{n} = \frac{100}{10} = 10$$

interval	O_i	$O_i - E_i$	$(O_i - E_i)^2$	E_i	$X_0^2 = \sum \frac{(O_i - E_i)^2}{E_i}$
1	8	-2	4	10	0.4
2	8	-2	4	10	0.4
3	10	0	0	10	0
4	9	-1	1	10	0.1
5	12	2	4	10	0.4
6	8	-2	4	10	0.4
7	10	0	0	10	0
8	14	4	16	10	1.6
9	10	0	0	10	0
10	11	1	1	10	0.1
Sum	100				3.4

$\therefore X_0^2 = 3.4$

from table A6, $X^2_{\alpha, n-1} = X^2_{0.05, 9} = 16.9$

$\therefore X_0^2 \leq X^2_{\alpha, n-1} = 3.4 \leq 16.9$

$\therefore$ Accepted null hypothesis

(6) Use chi-square Test with $\alpha = 0.05$ where $n = 10$, intervals of equal length. Sample data are given below:

0.34, 0.90, 0.25, 0.89, 0.87, 0.44, 0.12, 0.21, 0.46, 0.67,
0.83, 0.76, 0.79, 0.64, 0.70, 0.81, 0.94, 0.74, 0.22, 0.74,
0.96, 0.99, 0.77, 0.67, 0.56, 0.41, 0.52, 0.73, 0.99, 0.02
0.47, 0.30, 0.17, 0.82, 0.56, 0.05, 0.45, 0.31, 0.78, 0.05
0.79, 0.71, 0.23, 0.19, 0.82, 0.93, 0.65, 0.37, 0.39, 0.4
0.10, 0.17, 0.10, 0.46, 0.05, 0.66, 0.10, 0.42, 0.18, 0.49,
0.37, 0.51, 0.54, 0.01, 0.81, 0.28, 0.69, 0.34, 0.75, 0.49,
0.72, 0.43, 0.56, 0.97, 0.30, 0.94, 0.96, 0.58, 0.73, 0.05,
0.06, 0.39, 0.84, 0.24, 0.40, 0.64, 0.40, 0.19, 0.79, 0.62,
0.18, 0.26, 0.97, 0.88, 0.64, 0.47, 0.60, 0.11, 0.29, 0.78

Solution: given, $\alpha = 0.05$, $n = 10$, $N = 100$

$E_i = N/n = 100/10 = 10$

interval	O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$X_0^2 = \sum \frac{(O_i - E_i)^2}{E_i}$
0.01-0.10	10	10	0	0	0
0.11-20	8	10	-2	4	0.4
0.21-30	10	10	0	0	0
0.31-40	9	10	-1	1	0.1
0.41-50	12	10	2	4	0.4
0.51-60	8	10	-2	4	0.4
0.61-70	10	10	0	0	0
0.71-80	14	10	4	16	1.6
0.81-90	10	10	0	0	0
0.91-1.00	9	10	-1	1	0.1

$X_0^2 = 3$

from Table A6, $X_{\alpha, n-1}^2 = X_{0.05, 9}^2 = 16.9$

$\therefore X_0^2 \leq X_{\alpha, n-1}^2$

$3 \leq 16.9$

$\therefore$ Accepted Null hypothesis

(7) Using Auto corelation Test to test wether numbers are uniformly distributed with starting period 3rd, 8th, 13th & so on and largest integer number is 4. $Z_{\alpha/2} = 1.96$. the sample data are given below:

0.12, 0.01, 0.23, 0.28, 0.89, 0.31, 0.64, 0.28, 0.83, 0.93,
0.99, 0.15, 0.33, 0.35, 0.91, 0.41, 0.60, 0.27, 0.75, 0.88,
0.68, 0.49, 0.05, 0.43, 0.95, 0.58, 0.19, 0.36, 0.69, 0.87.

Solution: given, $i = 3$ (period starts from 3rd)

$m = 5$ (difference b/w periods i.e 8-3, 13-8 ...)

$M = 4$ (largest number)

$Z_{\alpha/2} = 1.96$

$$\hat{P}_{im} = \frac{1}{M+1}\left[\sum_{K=0}^{M} R_{i+Km} \, R_{i+(K+1)m}\right] - 0.25$$

$$= \frac{1}{4+1}\left[(0.23)(0.28) + (0.28)(0.33) + (0.33)(0.27) + (0.27)(0.05) + (0.05)(0.36)\right] - 0.25$$

$$= -0.1945$$

$$\sigma_{\hat{\rho}_{iM}} = \frac{\sqrt{13M+7}}{12(M+1)}$$

$$= \frac{\sqrt{13\times4+7}}{12(4+1)} = 0.1280$$

$$Z_0 = \frac{\hat{\rho}_{iM}}{\sigma_{\hat{\rho}_{iM}}} = \frac{-0.1945}{0.1280} = -1.5196$$

$\therefore$ $-Z_{\alpha/2} \le Z_0 \le Z_{\alpha/2}$

$-1.96 \le -1.5196 \le 1.96$

$\therefore$ Accepted Null Hypothesis

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(8). Generate 5 Random Variation from sample data 0.30, 0.20, 0.10, 0.50, 0.60 using Exponential distribution with mean = 1

%. Imp.

Solution : given $\lambda = 1$

i	1	2	3	4	5
R_i	0.30	0.20	0.10	0.50	0.60
X_i	0.3566	0.2231	0.1053	0.6931	0.9162

$$X_i = -\frac{1}{\lambda} \ln(1-R_i)$$

$$X_1 = -\frac{1}{1} \ln(1-0.30) = 0.3566$$

$$X_2 = -\frac{1}{1} \ln(1-0.20) = 0.2231$$

$X_3 = \frac{-1}{1} \ln(1-0.10) = 0.1053$

$X_4 = \frac{-1}{1} \ln(1-0.50) = 0.6931$

$X_5 = \frac{-1}{1} \ln(1-0.60) = 0.9162$

---

(9) Generate 5 Random Variation using Uniform Distribution technique with interval $0.3 \le x \le 2$.
consider sample data, 0.30, 0.25, 0.80, 0.75, 2.5

Solution: given, $a = 0.3$ $b = 2$

$$X_i = a + (b-a) R_1 \quad , i = 1 \ldots n$$

i	1	2	3	4	5
R_i	0.30	0.25	0.80	0.75	2.5
X_i	0.81	0	1.66	1.575	1

$X_1 = 0.3 + (2-0.3)\,0.3 = 0.81$

$X_2 = 0.25 < 0.3 = 0$

$X_3 = 0.3 \le 0.8 \le 2 \Rightarrow 0.3 + (2-0.3)\,0.80 = 1.66$

$X_4 = 0.3 \le 0.75 \le 2 \Rightarrow 0.3 + (2-0.3)\,0.75 = 1.575$

$X_5 = 2.5 > 2 \Rightarrow 1$

(10) Generate 6 random variation using Weibull Distribution with slope = 10 & shape = 2. the sample data are 0.10, 0.60, 0.50, 0.80, 0.20, 0.45

Solution: given $\alpha = 10$ (slope)
$\beta = 2$ (shape)

$$X_i = \alpha\left[-\ln(1-R_i)\right]^{1/\beta}$$

i	1	2	3	4	5	6
R_i	0.10	0.60	0.50	0.80	0.20	0.45
X_i	3.24	9.57	8.32	12.68	4.72	7.73

$$X_1 = 10 \times \left[-\ln(1-0.10)\right]^{1/2} = 3.24$$

$$X_2 = 10\left[-\ln(1-0.60)\right]^{1/2} = 9.57$$

$$X_3 = 10\left[-\ln(1-0.50)\right]^{1/2} = 8.32$$

$$X_4 = 10\left[-\ln(1-0.80)\right]^{1/2} = 12.68$$

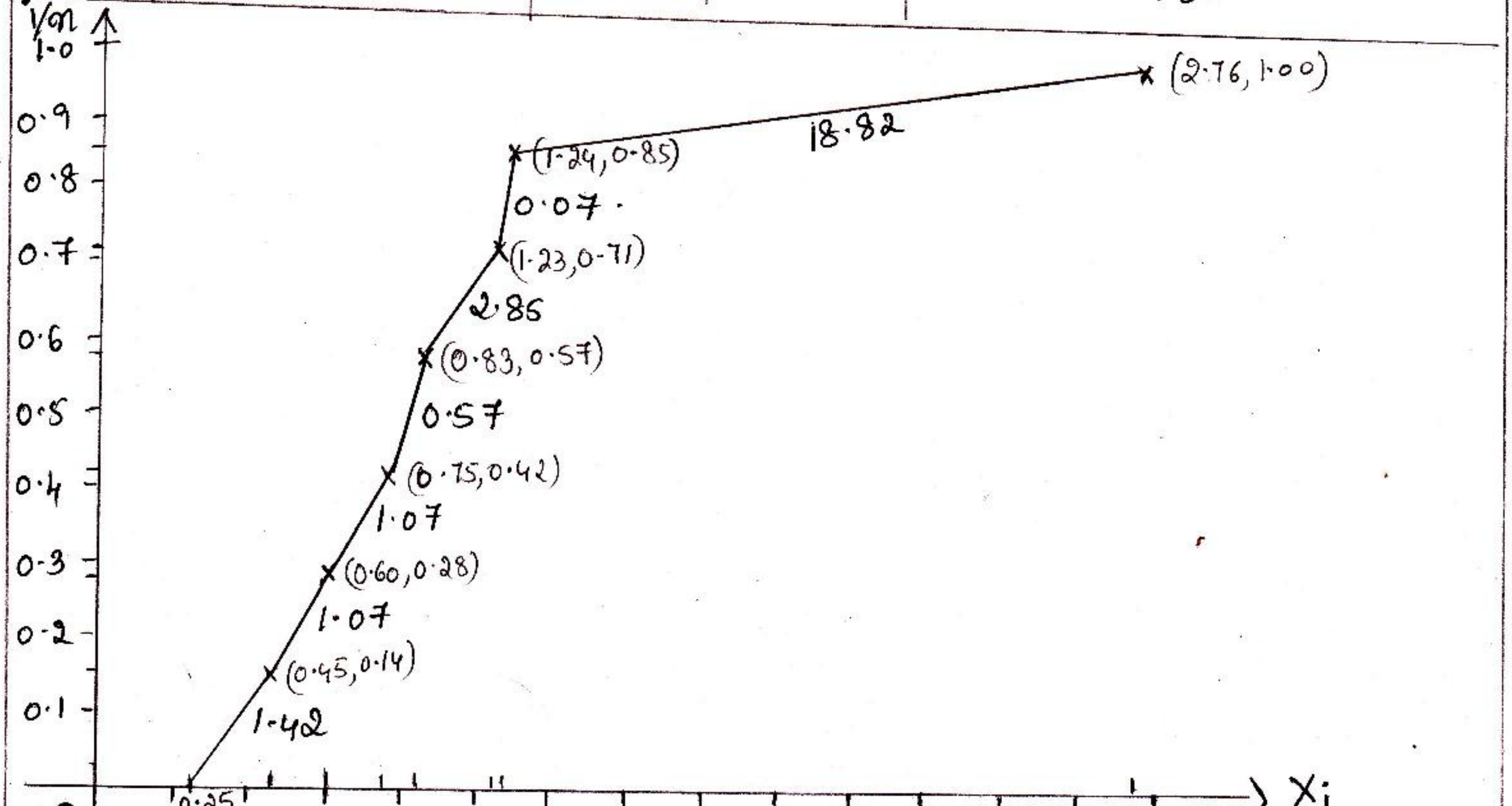
$$X_5 = 10\left[-\ln(1-0.20)\right]^{1/2} = 4.72$$

$$X_6 = 10\left[-\ln(1-0.45)\right]^{1/2} = 7.73$$

(11) consider the data : 0.83, 0.45, 1.23, 0.60, 0.75, 2.76, 1.24 with the probability = $1/n$ where $x_0 = 0.25$. Find slope of ith line segment using Emphirical Continuous distribution method. (without Frequency)

Solution:

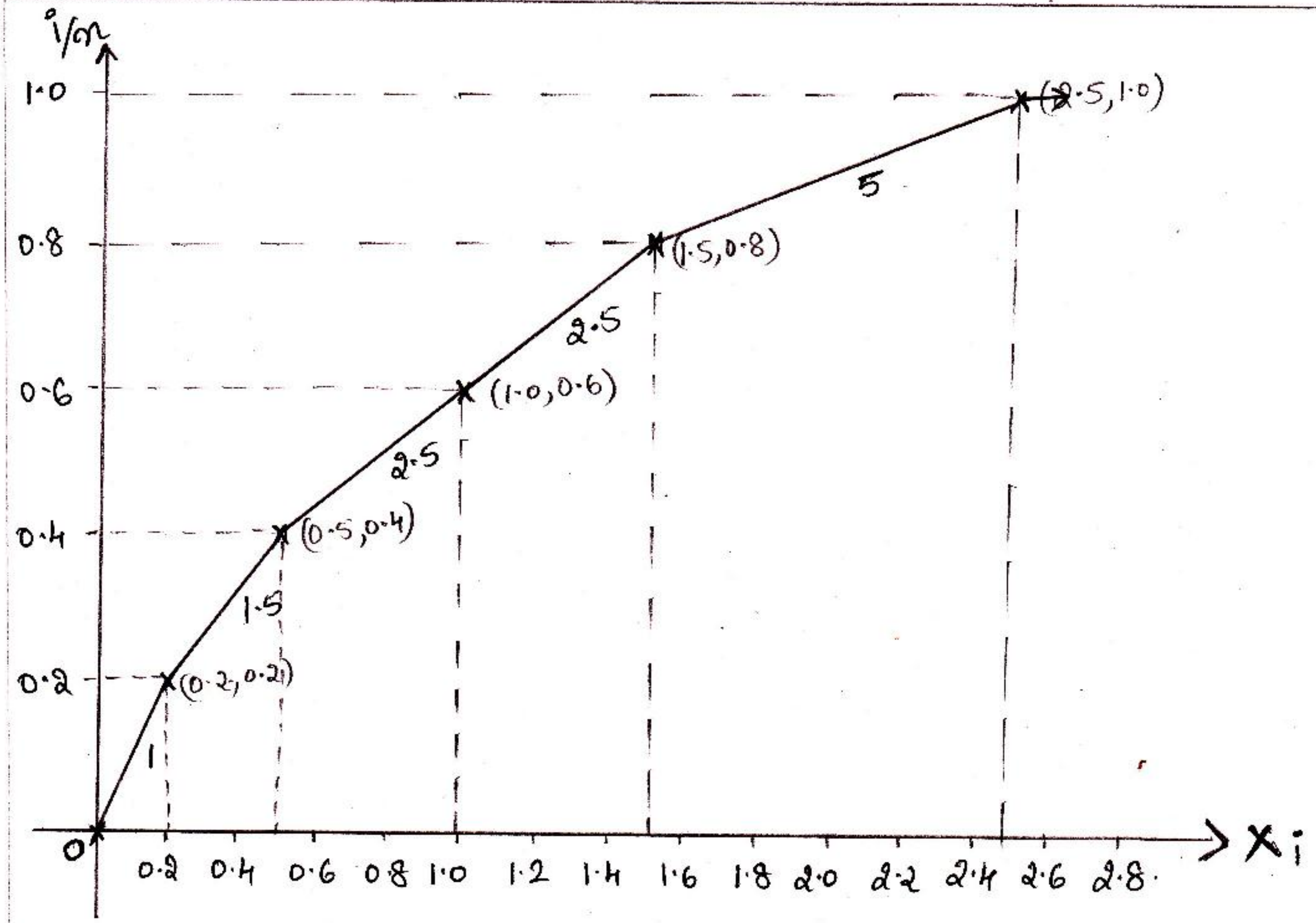
i	$x_{i-1} \leq x \leq x_i$	$1/n$	i/n	$a_i = \frac{x_i - x_{i-1}}{1/n}$
1	$0.25 \leq x \leq 0.45$	0.14	0.14	$a_i = \frac{0.45 - 0.25}{0.14} = 1.42$
2	$0.45 \leq x \leq 0.60$	0.14	0.28	1.07
3	$0.60 \leq x \leq 0.75$	0.14	0.42	1.07
4	$0.75 \leq x \leq 0.83$	0.14	0.57	0.57
5	$0.83 \leq x \leq 1.23$	0.14	0.71	2.85
6	$1.23 \leq x \leq 1.24$	0.14	0.85	0.07
7	$1.24 \leq x \leq 2.76$	0.14	1.00	18.82



(12) Consider the data 1.0, 0.5, 0.20, 1.5, 2.5 & Frequency are 31, 10, 25, 24, 30. Find the slope of ith line segment using Emphirical Continuous Distribution (with frequency)

Solution: consider $x_0 = 0.0$ & $c_0 = 0.0$

i	$x_{i-1} \leq x \leq x_i$	Frequency	Relative Frequency	Cumulative Probability (i/n)	$a_i = \frac{x_i - x_{i-1}}{c_i - c_{i-1}}$
1	$0.0 \leq x \leq 0.2$	25	0.25	0.2	$a_i = \frac{0.2-0.0}{0.2-0.0} = 1$
2	$0.2 \leq x \leq 0.5$	10	0.10	0.4	$\frac{0.5-0.2}{0.4-0.2} = 1.5$
3	$0.5 \leq x \leq 1.0$	31	0.31	0.6	$\frac{1.0-0.5}{0.6-0.4} = 2.5$
4	$1.0 \leq x \leq 1.5$	24	0.24	0.8	$\frac{1.5-1.0}{0.8-0.6} = 2.5$
5	$1.5 \leq x \leq 2.5$	30	0.30	1.00	$\frac{2.5-1.5}{1.0-0.8} = 5$



(13) 10m * Generate 3 poisson variate with mean 0.2. consider random numbers 0.4357, 0.4146, 0.8353, 0.9952, 0.8004.
vvimp

Solution: Steps: (2marks)

Step 0: Set $n=0$, $p=1$

2: $R_1 = 0.4357$ $\quad P = P \cdot R_1 = 1 \times 0.4357 = 0.4357$

$e^{-\alpha} = e^{-0.2} = 0.8187$

3: $P < e^{-\alpha} = 0.4357 < 0.8187$ ∴ Accept $N=0$

Step 1: $n=0$, $p=1$

2: $R_2 = 0.4146$ $\quad P = P \cdot R_2 = 1 \times 0.4146 = 0.4146$

3: $0.4146 < 0.8187$ ∴ Accept $N=0$

Step 1: $n=0$, $p=1$

2: $R_3 = 0.8353$ $\quad p = 1 \times 0.8353 = 0.8353$

3: $0.8353 < 0.8187$ ∴ Reject $n=1$ $\quad p = 0.8353$.

$n=1$, $p=0.8353$

Step 2: $R_4 = 0.9952$ $\quad p = P \cdot R_4 = 0.8353 \times 0.9952 = 0.8312$

3: $0.8312 < 0.8187$ ∴ Reject $n=2$

$n=2$, $p=0.8312$

Step 2: $R_5 = 0.8004$ $\quad p = 0.8312 \times 0.8004 = 0.6652$

3: $0.6652 < 0.8187$ ∴ Accept $N=2$

*table: (8 marks)

n	P	R_{n+1}	$P = P \cdot R_{n+1}$	$P < e^{-\alpha}$ A/R	$N = n$.
0	1	0.4357	0.4357	0.4357 < 0.8187 Accept	N=0
0	1	0.4146	0.4146	0.4146 < 0.8187 Accept	N=0
0	1	0.8353	0.8353	0.8353 < 0.8187 Reject	–
1	0.8353	0.9952	0.8312	0.8312 < 0.8187 Reject	–
2	0.8312	0.8004	0.6652	0.6652 < 0.8187 Accept	N=2

∴ the 3 poisson variates are N=0, N=0, N=2.

(14) Generate 5 poisson variate with mean = 0.2

Solution: Steps:

$e^{-\alpha} = e^{-0.2} = 0.8187$.

$n = 5$

*) If random numbers are not given, then the random numbers are selected so that it satisfies following conditions: If x is random number then,

$\frac{1}{4} \leq x \leq 1$ & $x \leq e^{-\alpha}$ then accepted, else reject.

Step 1: $n=0$, $p=1$

2: $R_1 = 0.3568$ $\quad p = p \times R_1 = 1 \times 0.3568 = 0.3568$

3: $0.3568 \leq 0.8187$ & $0.25 \leq 0.3568 \leq 1$ ∴ Accept N=0

Step 1) $n=0$ $\quad p=1$

2: $R_2 = 0.4123$ $\quad p = 1 \times 0.4123 = 0.4123$

3: $0.4123 \leq 0.8187$ & $0.25 \leq 0.4123 \leq 1$ ∴ Accept $N=0$

Step 1: $n=0$, $p=1$

2: $R_3 = 0.5067$ $p = 1 \times 0.5067 = 0.5067$

3: $0.5067 \leq 0.8187$ & $0.25 \leq 0.5067 \leq 1$ ∴ Accept $N=0$

$R_4 = 0.6818$ $p = 1 \times 0.6818 = 0.6818$

$0.6818 \leq 0.8187$ & $0.25 \leq 0.6818 \leq 1$ ∴ Accept $N=0$

Step 1: $n=0$, $p=1$

2: $R_5 = 0.7293$ $p = 1 \times 0.7293 = 0.7293$

3: $0.7293 \leq 0.8187$ & $0.25 \leq 0.7293 \leq 1$ ∴ Accept $N=0$

Table :

n	P	R_{n+1}	$P = P \cdot R_{n+1}$	$P \leq e^{-\alpha}$ A/R	$N = n$
0	1	0.3568	0.3568	$0.3568 \leq 0.8187$ Accept	N=0
0	1	0.4123	0.4123	$0.4123 \leq 0.8187$ Accept	N=0
0	1	0.5067	0.5067	$0.5067 \leq 0.8187$ Accept	N=0
0	1	0.6818	0.6818	$0.6818 \leq 0.8187$ Accept	N=0
0	1	0.7293	0.7293	$0.7293 \leq 0.8187$ Accept	N=0

∴ The 5 poisson variates are N=0, N=0, N=0, N=0, N=0

(15) Generate 5 gamma variate using Gamma distribution with slope, $\beta = 2.30$ & mean, $\theta = 0.4545$.
Consider the random numbers : 0.4357, 0.1806, 0.1508, 0.8353, 0.1202, 0.8004, 0.9550, 0.1460, 0.196, 0.234.

Solution :

given $\beta = 2.30$, $\theta = 0.4545$

Formula :

$$a = (2\beta - 1)^{1/2}$$

$$b = 2\beta - \ln 4 + 1/a$$

$$X = \beta \left[R_1 / (1 - R_1) \right]^a$$

$X \leq b - \ln(R_1^2 \cdot R_2)$ then Accept & $X = X/\beta\theta$ else reject.

Steps :

1: $a = (2 \times 2.3 - 1)^{1/2} = 1.8973$

$b = 2 \times 2.3 - \ln 4 + \frac{1}{1.8973} = 3.7407$

2: $R_1 = 0.4357 \quad R_2 = 0.1806$

3: $X = 2.30 \left[0.4357 / (1 - 0.4357) \right]^{1.8973} = 1.4080$

4: $1.4080 \leq 3.7407 - \ln(0.4357^2 \times 0.1806) = 7.1137$

∴ Accept hence $X = 1.4080 / 2.30 \times 0.4545 = 1.3469$

1: $a = 1.8973 \quad b = 3.7407$

2: $R_1 = 0.1508 \quad R_2 = 0.8353$

3: $X = 2.30 \left[0.1508 / (1 - 0.1508) \right]^{1.8973} = 0.0866$

4: $0.0866 \leq 3.7407 - \ln(0.1508^2 \times 0.8353)$

$0.0866 \leq 7.7042$ ∴ Accept

$X = 0.0866 / 2.3 \times 0.4545 = 0.0828$

1: $a = 1.8973$, $b = 3.7407$

2: $R_1 = 0.1202$ $R_2 = 0.8004$

3: $X = 2.3\left[0.1202/(1-0.1202)\right]^{1.8973} = 0.0526$

4: $0.0526 \leq 8.2005$ ∴ Accept

$x = 0.0526/2.3 \times 0.4545 = \boxed{0.0503}$

Step 1: $a = 1.8973$ $b = 3.7407$

2: $R_1 = 0.9550$ $R_2 = 0.1460$

3: $X = 756.9167$

4: $756.9167 \nleq 5.7569$ ∴ Reject

Step 2: $R_1 = 0.1960$ $R_2 = 0.2340$

3: $X = 0.1580$

4: $0.1580 \leq 8.4524$ ∴ Accept

$x = 0.1580/2.3 \times 0.4545 = \boxed{0.1511}$

table:

a	b	R_1 & R_2	X	$X < b - \ln(R_1^2 R_2)$ A / R	$x = X/\beta\theta$
1.8973	3.7407	$0.4357 = R_1$ $0.1806 = R_2$	1.4080	$1.4080 \leq 7.1137$ Accept	1.3469
1.8973	3.7407	$R_1 = 0.1508$ $R_2 = 0.8353$	0.0866	$0.0866 \leq 7.7042$ Accept	0.0828
1.8973	3.7407	$R_1 = 0.1202$ $R_2 = 0.8004$	0.0526	$0.0526 \leq 8.2005$ Accept	0.0503
1.8973	3.7407	$R_1 = 0.9550$ $R_2 = 0.1460$	756.916	$756.916 \leq 5.7569$ Reject	-
-	-	$R_1 = 0.196$ $R_2 = 0.234$	0.1580	$0.1580 \leq 8.4524$ Accept	0.1511

∴ The Gamma Variates are 1.34, 0.082, 0.0503, 0.1511